Fluid Dynamics

Chapter 4

Fluid Dynamics

- This chapter will deal with the application of Newton's law of motion for fluids.
- The law states that for a fluid particle (or control mass or a system) the rate of change of linear momentum equals the net force acting on the particle in an inertial coordinate system.

$$\sum f_{system} = \left(\frac{D(mv)}{Dt}\right)_{system}$$

If $f = (f_x \bar{i}, f_y \bar{j}, f_z \bar{k})$ and v = (u i, v j, w k), this vector equation can be decomposed into three scalar component equations,

$$f_x = \frac{D(mu)}{Dt}, f_y = \frac{D(mv)}{Dt} \text{ and } f_z = \frac{D(mw)}{Dt} \text{ for a fluid material element.}$$

Consider the rate of the extensive property B=m v with the intensive property b=v , Newton's law of motion will be

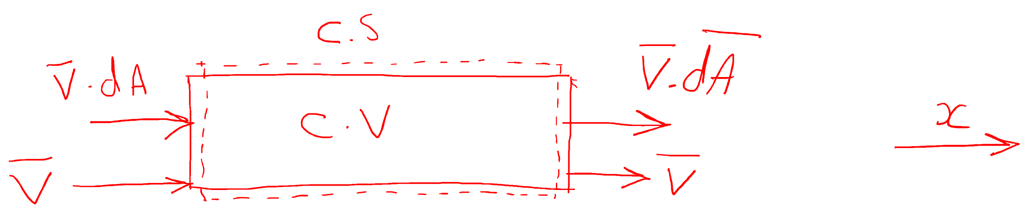
$$\sum \overline{F} = \frac{\partial}{\partial t}\iiint_{CV} \overline{V}\rho dV + \oiint_{CS} \overline{V}\rho(\overline{V}.\overline{dA})$$

$\overline{V}.$

Rate of change of fluid momentum

$$\sum \overline{F} = \frac{\partial}{\partial t}\iiint_{CV} \overline{V}\rho dV + \oint\!\!\oint_{CS} \overline{V}\rho(\overline{V}.\overline{dA})$$

- The left hand side is the net force acting on the particle or a system which coincides with the control volume at time t=0, thus the force calculated is the net force on the fluid with in the control volume.
- Note that the velocity vector appears twice in the control surface integral of the momentum equations.
- First it appears as the intensive property of linear momentum, second in the scalar product with the element of the control surface area vector $\overline{dA}$ giving the flow rate that transport linear momentum across the control surface.
- The same for the control volume continuity equation $\overline{dA}$ is outward normal from the area and the product $\overline{V}.\ \overline{dA}$ is positive for outflow and negative for inflow.



$\bar{V}$.

Rate of change of fluid momentum

$$\sum \overline{F} = \frac{\partial}{\partial t}\iiint_{CV} \overline{V} \rho dV + \oiint_{CS} \overline{V} \rho (\overline{V}.\overline{dA})$$

- The first term on the right hand side is the local rate of momentum change within the control volume. This term is usually zero for steady flow.

Rate of change of fluid momentum

- Example:

1. A stationary nozzle discharges a horizontal jet of water of a cross-sectional area 0.01 m^2 at a velocity 30 m/s. The jet strikes an inclined flat plate at 30^o to the horizontal. If the flow is incompressible, calculate the force normal to the plate for:

a) Fixed plate;

b) A plate moving with 10 m/s in the same direction as the jet.

* solution:-

a) For fixed plate:-

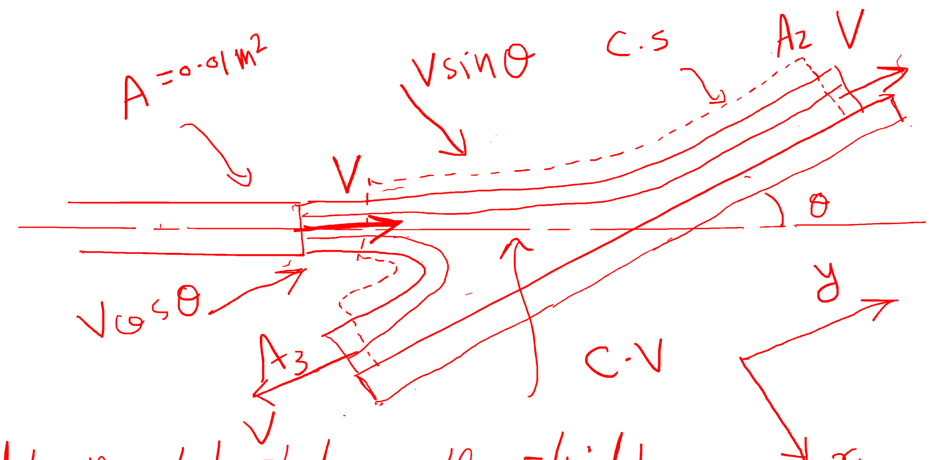
$$\sum F = \underbrace{\frac{\partial}{\partial t} \iiint_{C.V} \vec{V} \rho \, d\forall}_{\to 0 \text{ (steady)}} + \oint_{C.S} \vec{V} \rho (\vec{V} \cdot \vec{dA})$$

∵ normal components of forces are in x-direction

$$\therefore \sum F_x = (-\rho A_1 V)(V \sin\theta)$$

$$= -1000 * 0.01 * 30 \, (30 \sin 30)$$

$$F_x = -4500 \text{ N}$$ (the Force exerted by the Flat Plate on the Fluid)



Rate of change of fluid momentum

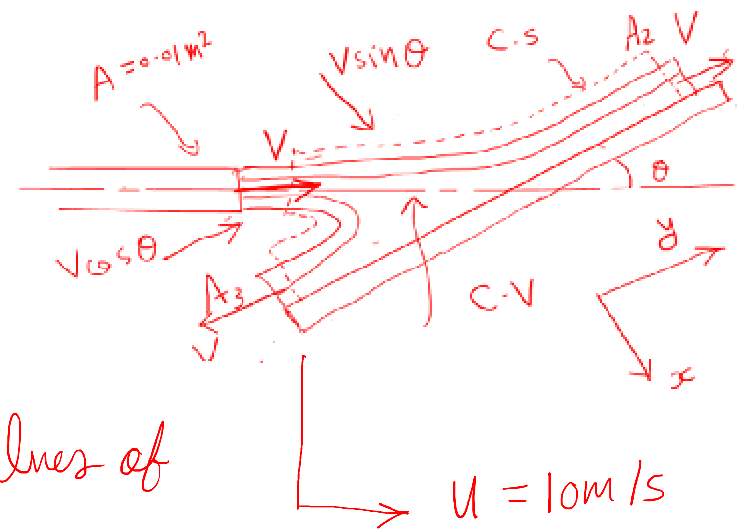
- Example continued:

b) in the case of moving the plate by $u = 10\ m/s$

$$F_x = -\rho A_1 (V-u)^2 \sin\theta$$

$$= -1000 * 0.01 * (30-10)^2 \sin 30$$

$$F_x = -2000\ N \quad (\text{Force on the Fluid})$$



Note: the Force on the plate have the same values of the previous cases but with positive signs.

Rate of change of fluid momentum

- Example:

- A stationary nozzle discharges a horizontal jet of water of a cross-sectional area 0.01 m^2 at a velocity 30 m/s. If the flow is incompressible, calculate the normal forces of the jet for both cases:

Case 1: The jet strikes vertical plate .

Case 2: The jet strikes hemispherical shape.

* solution:-

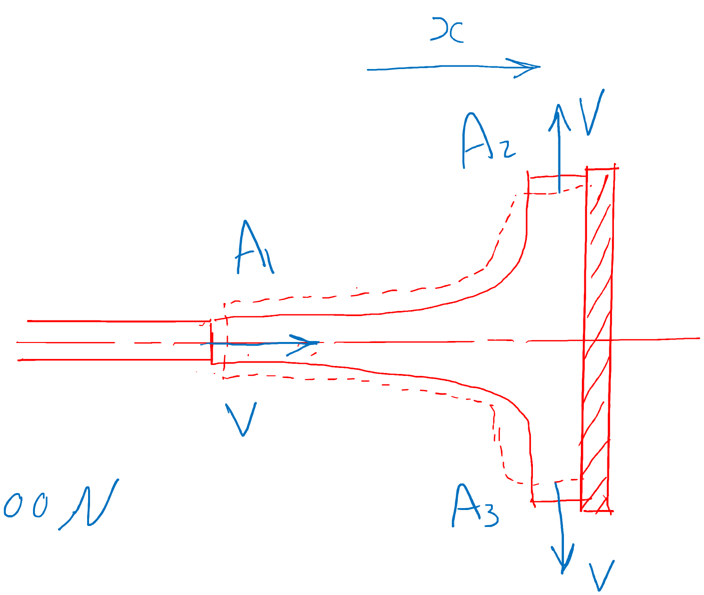
For case 1:-

$$F_x = (-\rho A_1 V)(V)$$

mass enters C.S (-ve) (velocity in positive direction)

$$F_x = -1000 * 0.01 * (30)^2 = -9000\,N$$

∴ Force of horizontal jet on vertical plate = 9000 N



Rate of change of fluid momentum

- Case 2: The jet strikes hemispherical shape

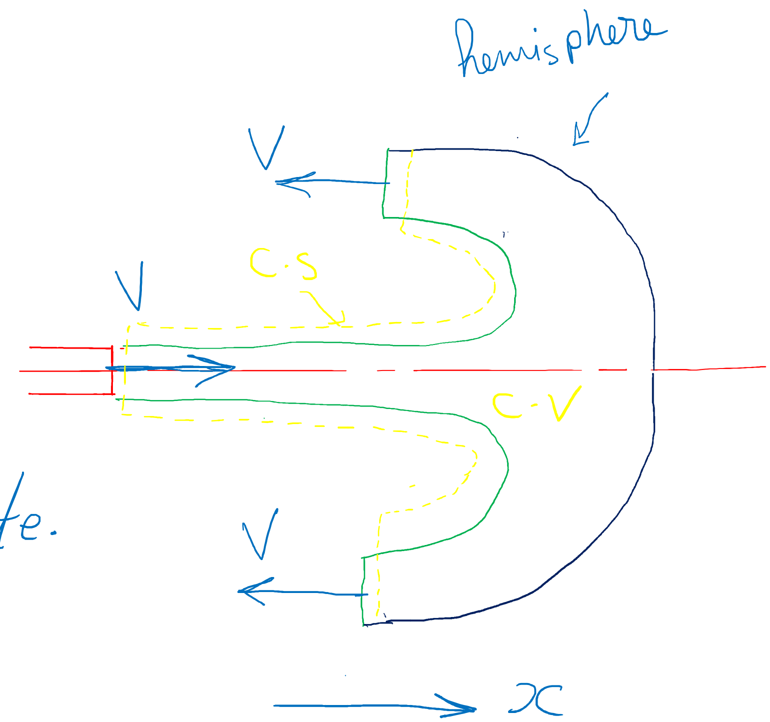
$$\sum F_x = (-\rho A V)(V) + (\rho A V)(-V)$$

$$= -2 \rho A V^2$$

$$= -2 * 1000 * 0.01 * (30)^2$$

$$= -18000 \text{ N}$$

$\therefore$ Force of jet on hemispherical shape equals twice the force on vertical flat plate.



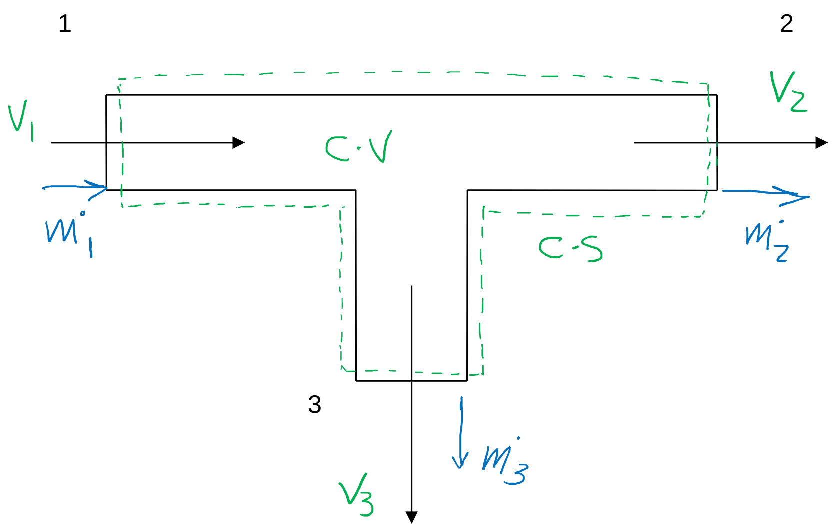
Rate of change of fluid momentum

- Example:

For the horizontal T connection water flows at the shown rates. Find the external force that must be applied to hold the T in place.

- $D_1 = D_2 = 15cm$, $Q_1 = 250$ l/s, $Q_2 = 100$ l/s, $D_3 = 10cm$, $P_1 = 100kPa$, $P_2 = 80kPa$, $P_3 = 70kPa$.

- Solution:



$\because Q = A V$

$$V_1 = \frac{Q_1}{A_1} = \frac{250 * 10^{-3}}{\frac{\pi}{4}(0.15)^2} = 14.14 \text{ m/s}$$

$$V_2 = \frac{Q_2}{A_2} = \frac{100 * 10^{-3}}{\frac{\pi}{4}(0.15)^2} = 5.65 \text{ m/s}$$

from continuity eq. $\Rightarrow Q_1 = Q_2 + Q_3$

$$Q_3 = 250 - 100 = 150 \text{ l/s}$$

$$V_3 = \frac{Q_3}{A_3} = \frac{150 * 10^{-3}}{\pi/4 (0.1)^2} = 19.098 \text{ m/s}$$

Rate of change of fluid momentum

- Example continued:

$$\sum \vec{F} = \underbrace{\frac{\partial}{\partial t}\iiint_{C.V} \vec{V}\rho\, d\forall}_{\to 0 \text{ (steady)}} + \oiint_{C.S} \rho \vec{V}\,(\vec{V}\cdot d\vec{A})$$

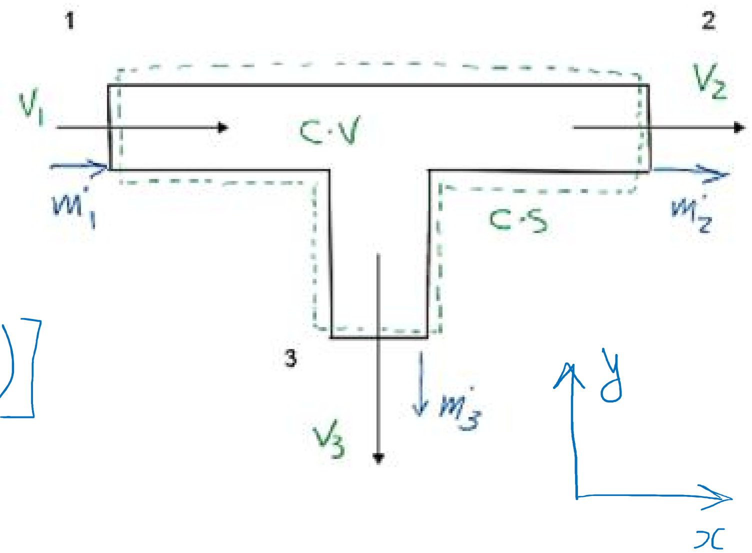
For forces in x-direction :-

$$F_x + P_1 A_1 - P_2 A_2 = \rho\left[(-Q_1)(V_1) + (Q_2)(V_2)\right]$$

$$F_x + (100 * 10^3)(\pi/4 * 0.15^2) - (80 * 10^3)\left(\frac{\pi}{4} * 0.15^2\right)$$

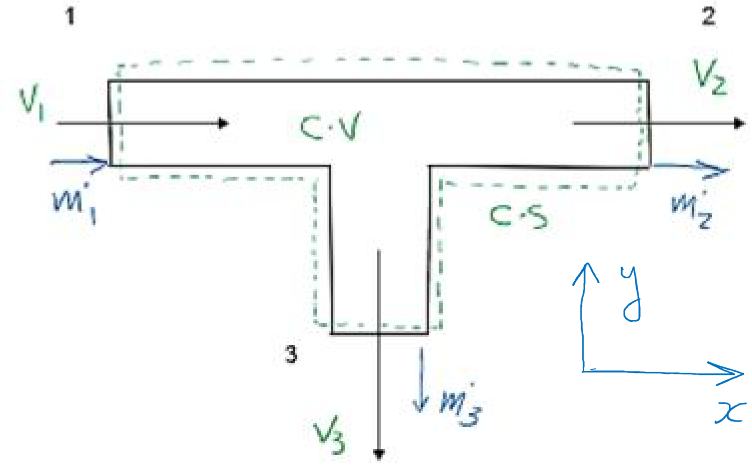
$$= 1000\left[(-0.25)(14.4) + (0.1)(5.65)\right]$$

$F_x = -3388.4$ N (The Force of T-Connection on the Fluid in x-direction)



Rate of change of fluid momentum

- Example continued:



* Forces in y-direction :-

$$F_y + P_3 A_3 = \rho (Q_3)(-V_3)$$

$$F_y + (70 * 10^3)(\pi/4 * 0.1^2) = 1000(0.15)(-19.09)$$

$$F_y = -3413.27 \text{ N}$$

$$F_R = \sqrt{F_x^2 + F_y^2} = \sqrt{(3388)^2 + (3413)^2}$$

$$F_R = 4809 \text{ N}$$

Rate of change of fluid momentum

- Example:

- A pipe bend tapers from a diameter of d_1 of 500 mm at inlet to a diameter of d_2 of 250 mm at outlet and turns the flow through an angle θ of 45º. Measurements of pressure at inlet and outlet show that the pressure P_1 at inlet is 40 kN/m² and the pressure P_2 at outlet is 23 kN/m². If the pipe is conveying oil which has density of 850 kg/m³, calculate the magnitude and direction of the resultant force on the bend when the oil is flowing at the rate of 0.45 m³/s. the bend is in a horizontal plane.

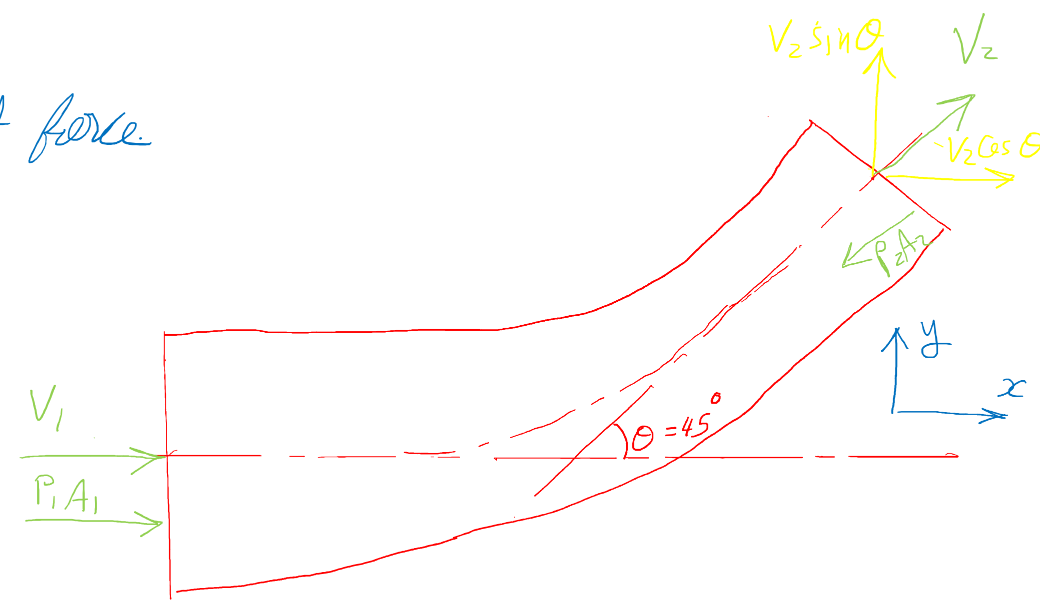
* Required:-

the magnitude and the direction of the restultant force

* solution:-

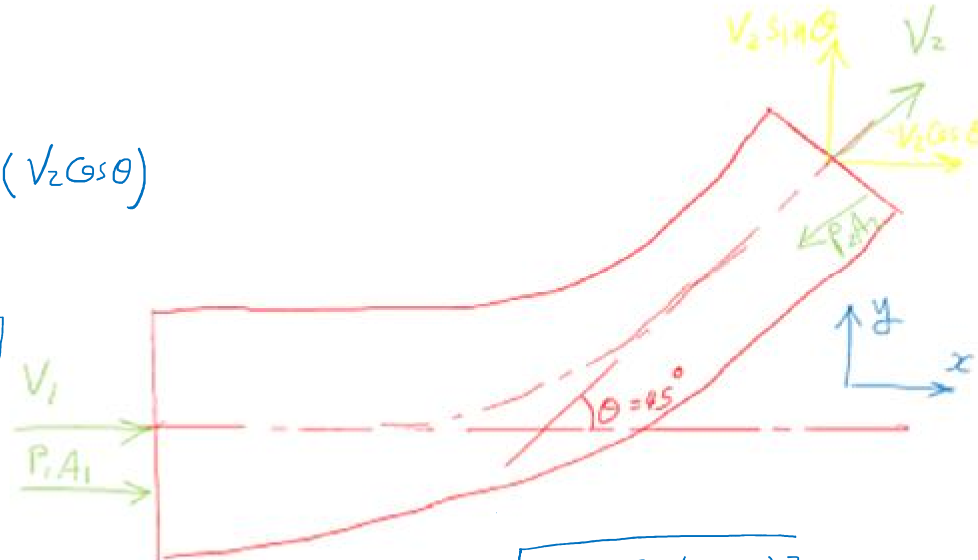
$$V_1 = \frac{Q}{A_1} = \frac{0.45}{\frac{\pi}{4}(0.5)^2} = 2.3\ m/s$$

$$V_2 = \frac{Q}{A_2} = \frac{0.45}{\frac{\pi}{4}(0.25)^2} = 9.167\ m/s$$



Rate of change of fluid momentum

- Example continued:



* Forces in x-direction:-

$$F_x + P_1A_1 - P_2A_2\cos\theta = (-\rho A_1V_1)(V_1) + (\rho A_2V_2)(V_2\cos\theta)$$

$$F_x + \left(40*10^3*\frac{\pi}{4}0.5^2\right) - \left(23*10^3*\frac{\pi}{4}0.25^2\cos 45\right)$$

$$= 1000*\left[\left(-\frac{\pi}{4}*0.5^2*2.29^2\right) + \left(\frac{\pi}{4}*0.25^2*9.16^2\cos 45\right)\right]$$

$$F_x = -5173\ N$$

* Forces in y-direction:-

$$F_y - P_2A_2\sin\theta = (\rho A_2V_2)(V_2\sin\theta)$$

$$F_y - \left(23*10^3*\frac{\pi}{4}0.25^2*\sin 45\right) = 1000*\frac{\pi}{4}(0.25)^2(9.16)^2\sin 45$$

$$F_y = 3710.6\ N$$

$$R = \sqrt{(5173)^2 + (3710)^2}$$

$$R = 6366\ N$$

$$\tan\theta = \frac{3710.6}{5173}$$

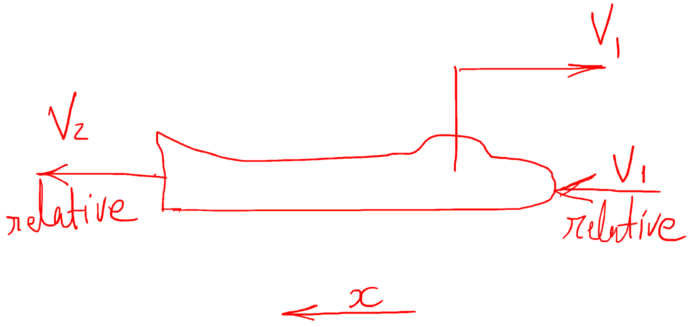
$$\theta = 35^\circ$$

Application of the Momentum Equation to Propulsion Engines

- These engines create jets. The jet exerts a thrust force, and this force is known as the propelling force.
- We will apply the momentum conservation principle to develop relationship for the net thrust force and other flow parameters.
- These engines can be classified as either air- breathers (with inlet mass of air into the control volume) such as turbojet turbofan, turboprop, ramjet, and pulsejet or non-air-breathers (no inlet mass of air) called rockets.

- Consider the shown airplane traveling at a velocity v_1 through still air. The air is taken into the engine where it is burned with small amount of fuel.
- The products of mass m_2 are ejected at a velocity v_2 relative to the airplane.
- Apply the momentum equation considering the flow to be steady, local rate of change of momentum is zero and the exchange of momentum across the surface of a control volume moving with a speed v_1 of the airplane gives;

$$f_x = m_2 v_2 - m_1 v_1$$



Application of the Momentum Equation to Propulsion Engines

- If the mass of fuel is neglected w r t the mass of air then $m_2=m_1$.
- The useful work is F v_1.
- The exhaust kinetic energy per unit time is $0.5\dot{m}_2 (v_2-v_1)^2$.
- where $(v_2-v_1)$is the absolute velocity of the exhaust gases.
- One expression of propulsive efficiency

$$h_p=\frac{Fv_1}{Fv_1+0.5m_2(v_2-v_1)^2}$$

Rate of change of fluid momentum

- Example:

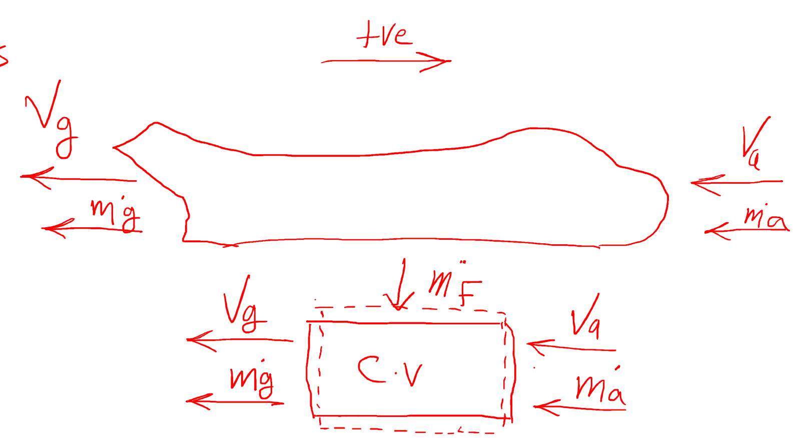
1. A jet engine consumes 1 kg of fuel for each 40 kg of air passing through the engine. The fuel consumption is 1.1 kg/s when the aircraft is traveling in still air at a speed of 200 m/s. The velocity of the gases which are discharged at atmospheric pressure from the tailpipe is 700 m/s relative to the engine. Calculate:

- i) The thrust of the engine,
- ii) The work done per second, and iii) the efficiency.

* Givens:-

$A/F = 40\ kg.a/kg.F$, $\dot{m}_F = 1.1\ kg/sec$, $V_a = 200\ m/s$

$V_g = 700\ m/sec$



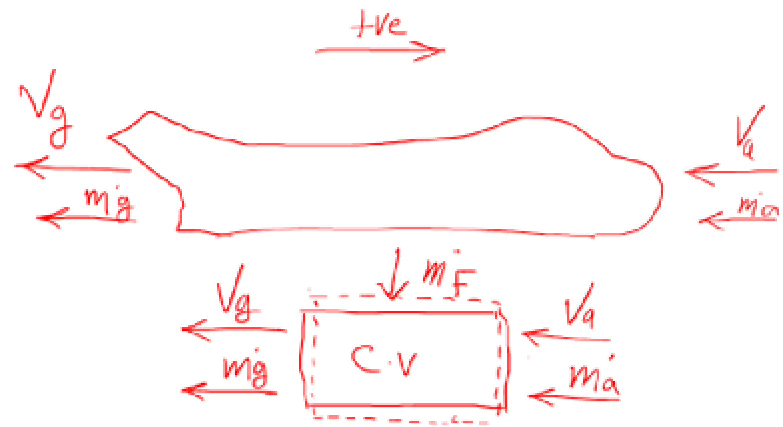
* Solution:-

$\dot{m}_a = 40 * 1.1 = 44\ kg/sec$

$\dot{m}_g = 44 + 1.1 = 45.1\ kg/sec$

Rate of control of fluid momentum

- Example continued:



Note: for moving C.V with constant speed, all used velocities in the equation should be relative velocities.

V_a : The air velocity relative to the jet engine.

V_g : The gases velocity relative to the jet engine.

i) $F_{th} = (-\dot m_a)(-V_a) + (\dot m_g)(-V_g)$

$= 44 * 200 - 45.1 * 700 = -22770\ N$

$\therefore \boxed{F_{th} = 22770\ N}$

ii) Work done per second (Power) $= F_{th} * V_a$

$= 22770 * 200 = 4.554 * 10^3\ kW$

iii) $\eta_{propulsive} = \dfrac{\text{Power}}{\text{Power} + \text{losses}} = \dfrac{F_{th} * V_a}{F_{th} * V_a + \frac{1}{2}\dot m_g (V_g - V_a)^2} * 100 = \dfrac{4.554 * 10^3 * 100}{4.554 * 10^3 + \frac{1}{2} * 45.1 * (700 - 200)^2}$

$\eta_{propulsive} = 44.68\ \%$

Rate of change of fluid momentum

- Example:

- A turbojet engine uses 22.7 kg/s of air with a fuel-to- air mass ratio of 0.0191. The flight speed is 269 m/s and the gases are exhausted at 514 m/s. The exhaust pressure is 0.51 bar and exit area 0.22 m^2. The fuel calorific value equals 39.4 MJ/kg. Calculate the thrust and net power. Calculate the propulsive and thermal efficiencies.

Solution

The inlet mass flow rate of air m_1=22.7 kg/s

Fuel-to-air mass ratio f/A = 0.0191

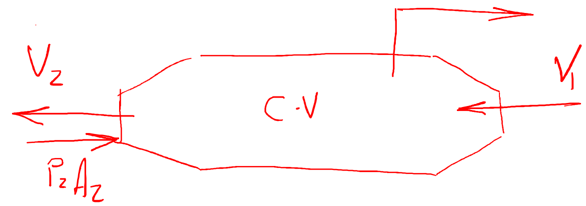
The exit mass flow rate of gases

m_2=22.7(1+f) = 23.13 kg/s

The flight speed v_1=269 m/s.

Exit pressure p_2=0.51x10^5 N/m^2

Relative velocity of exhaust gases, v_2=514 m/s



Absolute velocity of exhaust gases

$V_{2\ abs}$= 514- 269= 245 m/s

Rate of change of fluid momentum

- Example continued:

Consider the shown control volume moving with the flight speed v_1 and apply the momentum equation,

$F - P_2A_2 = m_2v_2 - m_1v_1$ with all velocities relative to the control volume.

$F=23.13(514)-22.7(269)+0.51(10^5)(0.22)$

$= \quad 17004$ N.

- The propulsive power = Fv_1 = 17004 (269)(10^{-6})=4.574 MW
- The minimum losses =$0.5m_2V_2{}^2{}_{abs}$ =0.5 (23.13)($245)^2$=694 kW

- $Propulsive\ efficiency(\eta_b) = \frac{Useful\ power}{Useful\ power+min.losses} = \frac{4574}{(4574+694)} * 100 = 86.8\%$
- $Thermal\ efficiency(\eta_{th}) = \frac{Useful\ propulsive\ power}{heat\ added} = \frac{4574}{22.7(0.019)(39400)} * 100 = 27.8\%$

Angular Momentum (Moment of Momentum) Equation.

- In this section we discuss the relation between the net moment on the fluid element and the rate of change of angular momentum.

- If the moment of each term of the integral momentum equation is taken about the origin 0, the moment is the vector product of the vector r and the force vector.

$$\sum \bar{r} x \bar{f} = \frac{\partial}{\partial t} \iiint_{c.v.} \bar{r} x \bar{V} \rho \, dV + \oiint_{c.s.} (\bar{r} x \bar{V}) \rho \, (\bar{V} . \bar{ds})$$

- It states that the net torque on the fluid equals the sum of the local rate of change of angular momentum within the control volume and the net flow rate of angular momentum from the control surface. For steady flow the first term on the right-hand side vanishes.

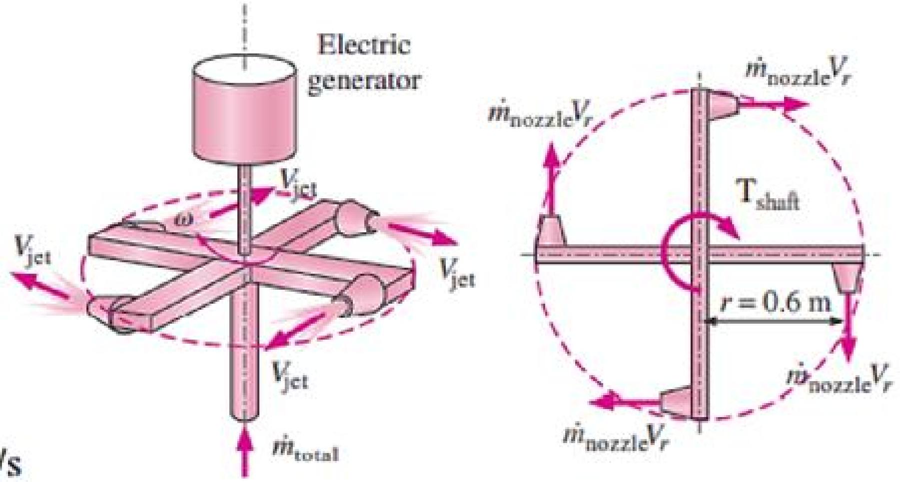
Angular Momentum (Moment of Momentum) Equation

- Example: (Power Generation from a Sprinkler System)

- A large lawn sprinkler with four identical arms is to be converted into a turbine to generate electric power by attaching a generator to its rotating head, as shown in the figure. Water enters the sprinkler from the base along the axis of rotation at a rate of 20 L/s and leaves the nozzles in the tangential direction. The sprinkler rotates at a rate of 300 rpm in a horizontal plane. The diameter of each jet is 1 cm, and the normal distance between the axis of rotation and the center of each nozzle is 0.6 m. Estimate the electric power produced.

SOLUTION

The conservation of mass equation for this steady-flow system is $\dot{m}_1 = \dot{m}_2 = \dot{m}_{total}$. Noting that the four nozzles are identical, we have $\dot{m}_{Nozzle} = {}^{\dot{m}_{total}}/_{4}$ or $\dot{Q}_{Nozzle} = {}^{\dot{Q}_{total}}/_{4}$ since the density of water is constant. The average jet exit velocity relative to the nozzle is



$$V_{jet} = \frac{\dot{Q}_{Nozzle}}{A_{jet}} = \frac{5 \text{ L/s}}{[\pi(0.01 \text{ m})^2/4]}\left(\frac{1 \text{ m}^3}{1000 \text{ L}}\right) = 63.66 \text{ m/s}$$

Angular Momentum (Moment of Momentum) Equation

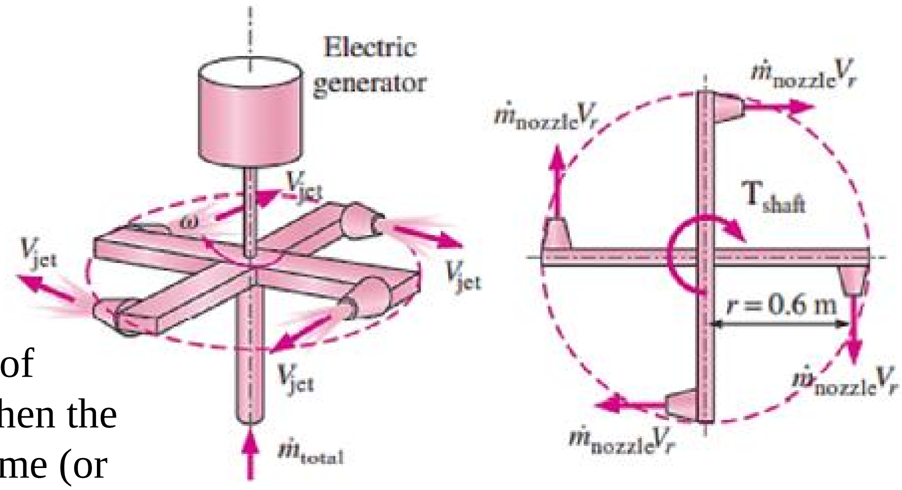
- Example: (Power Generation from a Sprinkler System)

SOLUTION continued

The angular and tangential velocities of the nozzles are

$$\omega = 2\pi\dot{n} = 2\pi(300 \text{ rev/min})\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 31.42 \text{ rad/s}$$

$$V_{\text{nozzle}} = r\omega = (0.6 \text{ m})(31.42 \text{ rad/s}) = 18.85 \text{ m/s}$$



That is, the water in the nozzle is also moving at a velocity of 18.85 m/s in the opposite direction when it is discharged. Then the average velocity of the water jet relative to the control volume (or relative to a fixed location on earth) becomes

$$V_r = V_{\text{jet}} - V_{\text{nozzle}} = 63.66 - 18.85 = 44.81 \text{ m/s}$$

$$-T_{\text{shaft}} = -4r\dot{m}_{\text{nozzle}}V_r \qquad \text{or} \qquad T_{\text{shaft}} = r\dot{m}_{\text{total}}V_r$$

$$T_{\text{shaft}} = r\dot{m}_{\text{total}}V_r = (0.6 \text{ m})(20 \text{ kg/s})(44.81 \text{ m/s})\left(\frac{1 \text{ N}}{1 \text{ kg}\cdot\text{m/s}^2}\right) = 537.7 \text{ N}\cdot\text{m}$$

$$\dot{W} = 2\pi\dot{n}T_{\text{shaft}} = \omega T_{\text{shaft}} = (31.42 \text{ rad/s})(537.7 \text{ N}\cdot\text{m})\left(\frac{1 \text{ kW}}{1000 \text{ N}\cdot\text{m/s}}\right) = \mathbf{16.9 \text{ kW}}$$

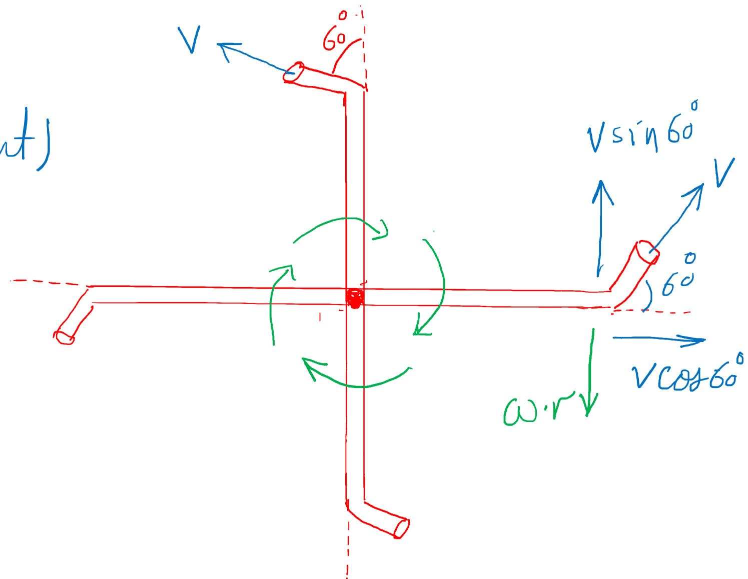
Angular Momentum (Moment of Momentum) Equation

• Example:

Water enters the shown sprinkler at volume flow rate of 2.5 l/s at the axis of rotation and flow out through four arms. Each arm has a radius of 0.3 m and nozzle 1 cm diameter at its end with axis at 60° angle with the tangent. Determine: i) The power developed if the rotational speed was 120 rpm,

ii) The maximum speed of rotation (no friction) at the axis of rotation,

iii) Torque required holding the sprinkler stationary.



* Givens :-

$Q_{total} = 2.5\ l/sec$, $r = 0.3\ m$, $d = 1\ cm$, $\theta = 60°$ (with the tangent)

$N = 120\ r.p.m$, $n = 4$ arms

* Required :-

i) the developed power   ii) N_{max} (no friction)

iii) T_{max} at $(N = 0)$

Angular Momentum (Moment of Momentum) Equation

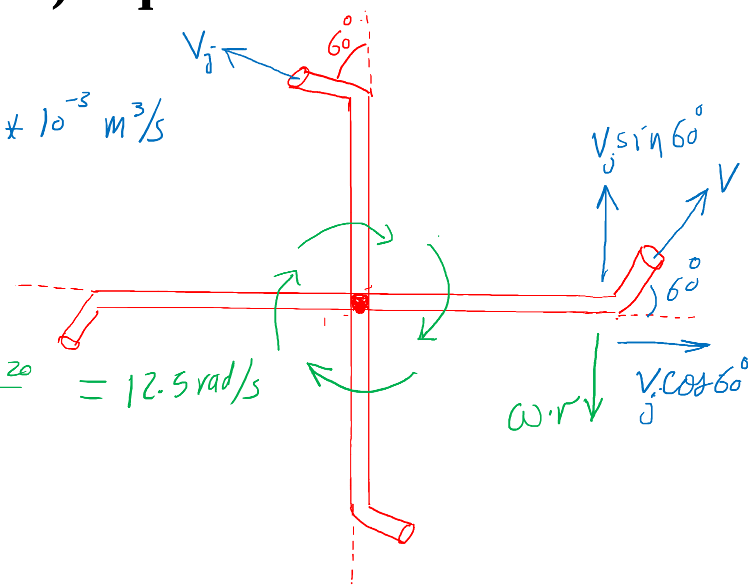
- Solution:

$$Q_{jet} = \frac{Q_{total}}{4} = \frac{2.5}{4} \Rightarrow Q_{jet} = 0.625\ l/s = 0.625 * 10^{-3}\ m^3/s$$

$$A_{jet} = \frac{\pi}{4} d_j^2 = \frac{\pi}{4}(0.01)^2 = 0.0000785\ m^2$$

$$V_j = \frac{Q_j}{A_j} = \frac{0.625 * 10^{-3}}{0.0000785} \Rightarrow V_j = 7.95\ m/s$$

$$T = 4\dot{m}_{jet}\,(V_j \sin\theta - \omega r)\, r \quad , \quad \omega = \frac{2\pi N}{60} = \frac{2\pi * 120}{60} = 12.5\ rad/s$$



i) $T = 4 * 1000 * 0.625 * 10^{-3}\,(7.95 \sin 60 - 12.5 * 0.3) * 0.3$

$\therefore\ T = 2.35\ N$

$Power = T \cdot \omega = 2.35 * 12.5 \Rightarrow \boxed{Power = 29.39\ W}$

ii) for no-friction case $(T = 0)$

$\therefore\ 0 = 4 * 0.625\,(7.95 \sin 60 - 0.3\,\omega_{max}) * 0.3 \Longrightarrow \omega_{max} = 23\ rad/sec \Longrightarrow \boxed{N_{max} = 219.1\ rpm}$

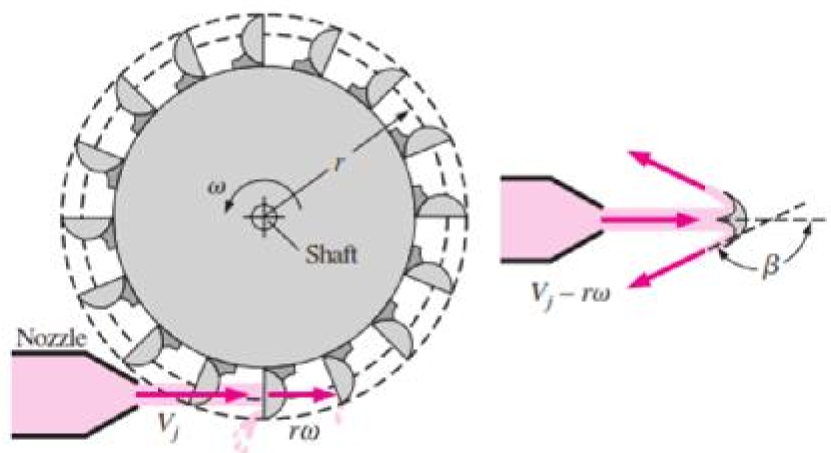
iii) for T_{max} at $(N = 0)$

$\therefore\ T_{max} = 4 * 0.625\,(7.95 \sin 60 - 0) * 0.3 \Longrightarrow \boxed{T_{max} = 5.16\ N.m}$

Angular Momentum (Moment of Momentum) Equation

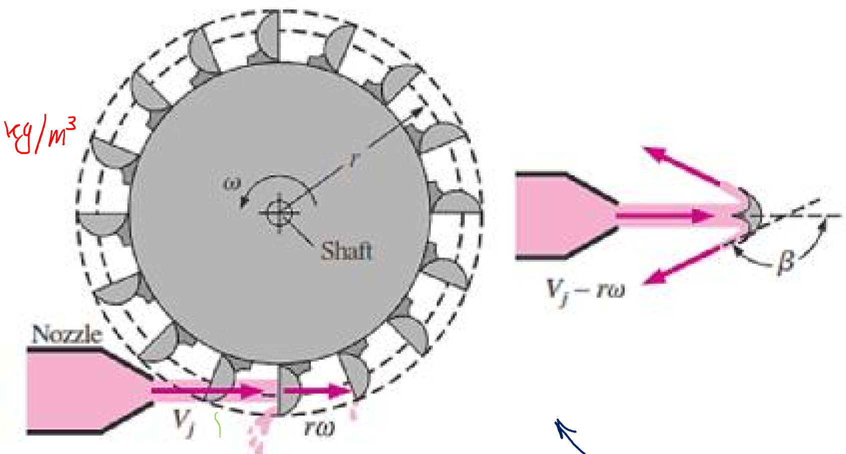
- Example:

Pelton wheel turbines are commonly used in hydroelectric power plants to generate electric power. In these turbines, a high-speed jet at a velocity of V_j impinges on buckets, forcing the wheel to rotate. The buckets reverse the direction of the jet, and the jet leaves the bucket making an angle b with the direction of the jet, as shown in the figure. Show that the power produced by a Pelton wheel of radius r rotating steadily at an angular velocity of w is $\dot{W}_{shaft} = \rho\omega r Q(V_j - \omega r)(1 - \cos\beta)$, where ρ is the density and Q is the volume flow rate of the fluid. Obtain the numerical value for $\rho = 1000\ \text{kg}/m^3$, r=2 m, Q = 10 m^3/s, N=150 rpm, B=160^o, and V_j=50m/s.



Angular Momentum (Moment of Momentum) Equation

- Solution:



* Givens :-

$V_j = 50\, m/s$, $r = 2m$, $Q = 10\, m^3/s$, $N = 150\, rpm$, $\rho = 1000\, kg/m^3$

$\omega = \frac{2\pi N}{60} = \frac{2\pi * 150}{60} = 15.7\, rad/s$

* solution:-

$-F = (-\dot{m}) V_r + (\dot{m})(-V_r \cos(180-\beta))$

$-F = \dot{m} V_r \cos\beta - \dot{m} V_r$

$\therefore T = F \cdot r$

$\therefore T = r\dot{m}V_r - r\dot{m}V_r \cos\beta$

$= r\dot{m}V_r (1-\cos\beta)$

$\therefore T = r\dot{m}(V_j - \omega r)(1-\cos\beta)$

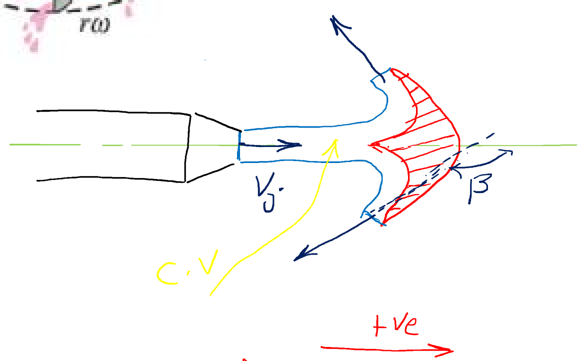
$\dot{W}_{shaft} = T \cdot \omega$

$\therefore \dot{W}_{shaft} = \rho \omega r Q (V_j - \omega r)(1-\cos\beta)$

$\therefore \dot{W}_{shaft} = 1000 * 15.7 * 2 * 10 * (50 - 15.7*2)(1-\cos 160)$

$\therefore \dot{W}_{shaft} = 11.3 * 10^6\ W$

$= 11.3\ MW$



V_r : relative velocity

$V_r = V_j - \omega r$